Solve $2^x + x - 2 = 0$

A. Preliminary investigations

Let $f(x) = 2^x + x - 2$

$f'(x) = \ln(2) . 2^x + 1 > 0$

Therefore $f(x)$ is an increasing function.

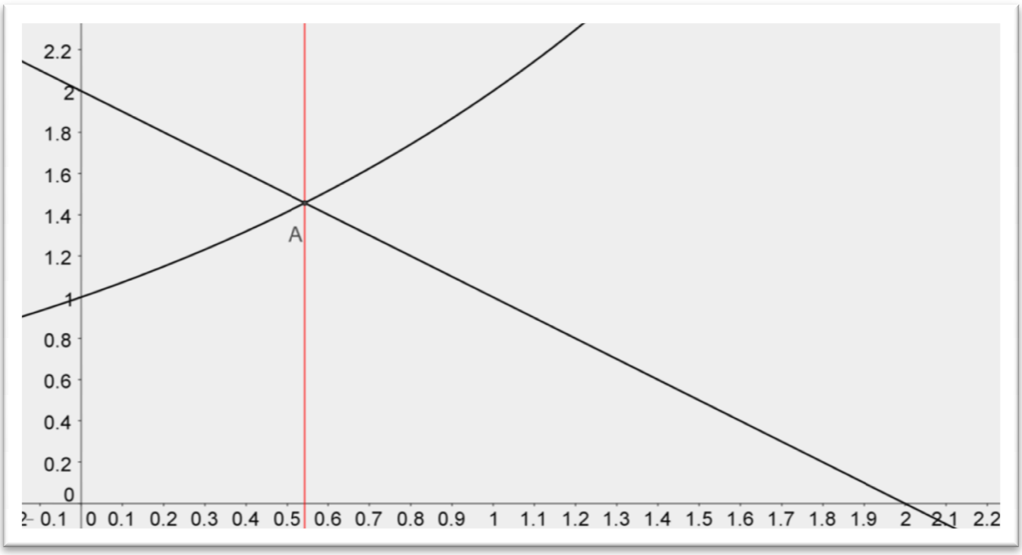
$f(0) = 2^0 + 0 - 2 = -1 < 0$

$f(1) = 2^1 + 1 - 2 = 1 > 0$

Since there is a sign change, we have one (or more) real root x_0, such that $0 < x_0 < 1$.

B. Graphical method

$$2^x + x - 2 = 0 \Leftrightarrow \begin{cases} y = 2^x \\ y = 2 - x \end{cases}$$



The graphs show that there is a root x_0, such that $0.4 < x_0 < 0.5$.

C. Numerical analysis (See Support document : [Numeral method.xlsx])

(1) Bisection method

The Bisection Method

Given a continuous function f(x) in an interval [a , b] and f(a) and f(b) are of opposite signs (There is at least one zero crossing within the interval.)

1. Find the mid-point $c = \frac{a+b}{2}$.
2. Find $f(c)$.
3. Replace $(a, f(a))$ or $(b, f(b))$ with $(c, f(c))$ so that there is zero crossing in the new interval.
4. Repeat 1 – 3 until $f(c)$ is sufficiently close to zero, and c is the approximate root.

We use EXCEL to get the iterations as below:

Step	a	Mid-pt c=(a+b)/2	b	f(a)	f(c)	f(b)
1	0	0.5	1	-1	-0.085786438	1
2	0.5	0.75	1	-0.085786438	0.431792831	1
3	0.5	0.625	0.75	-0.085786438	0.167210825	0.431792831
4	0.5	0.5625	0.625	-0.085786438	0.039326146	0.167210825
5	0.5	0.53125	0.5625	-0.085786438	-0.023569193	0.039326146
6	0.53125	0.546875	0.5625	-0.023569193	0.007792794	0.039326146
7	0.53125	0.5390625	0.546875	-0.023569193	-0.007909504	0.007792794
8	0.5390625	0.54296875	0.546875	-0.007909504	-6.36956E-05	0.007792794
9	0.54296875	0.544921875	0.546875	-6.36956E-05	0.003863212	0.007792794
10	0.54296875	0.543945313	0.544921875	-6.36956E-05	0.001899424	0.003863212
11	0.54296875	0.543457031	0.543945313	-6.36956E-05	0.000917781	0.001899424
12	0.54296875	0.543212891	0.543457031	-6.36956E-05	0.000427022	0.000917781
13	0.54296875	0.54309082	0.543212891	-6.36956E-05	0.000181658	0.000427022
14	0.54296875	0.543029785	0.54309082	-6.36956E-05	5.89798E-05	0.000181658
15	0.54296875	0.542999268	0.543029785	-6.36956E-05	-2.35821E-06	5.89798E-05
16	0.542999268	0.543014526	0.543029785	-2.35821E-06	2.83107E-05	5.89798E-05
17	0.542999268	0.543006897	0.543014526	-2.35821E-06	1.29762E-05	2.83107E-05
18	0.542999268	0.543003082	0.543006897	-2.35821E-06	5.30901E-06	1.29762E-05
19	0.542999268	0.543001175	0.543003082	-2.35821E-06	1.4754E-06	5.30901E-06
20	0.542999268	0.543000221	0.543001175	-2.35821E-06	-4.41403E-07	1.4754E-06
21	0.543000221	0.543000698	0.543001175	-4.41403E-07	5.16999E-07	1.4754E-06
22	0.543000221	0.54300046	0.543000698	-4.41403E-07	3.77977E-08	5.16999E-07
23	0.543000221	0.54300034	0.54300046	-4.41403E-07	-2.01803E-07	3.77977E-08
24	0.54300034	0.5430004	0.54300046	-2.01803E-07	-8.20026E-08	3.77977E-08
25	0.5430004	0.54300043	0.54300046	-8.20026E-08	-2.21025E-08	3.77977E-08
26	0.54300043	0.543000445	0.54300046	-2.21025E-08	7.84762E-09	3.77977E-08
27	0.54300043	0.543000437	0.543000445	-2.21025E-08	-7.12742E-09	7.84762E-09

In each step, we discard half of the interval of the previous step. The method is slow and we get the root of $\mathbf{2^x + x - 2 = 0}$ to be $x \approx \underline{\underline{\mathbf{0.5430004}}}$, to 7 decimal places in 27 steps.

(2) Newton-Raphson Method

Newton-Raphson Method

Choose x_0 as initial value and use the formula: $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

repeatedly to get the solution until we get a sufficiently small error. (We do not discuss error estimation here and in later methods)

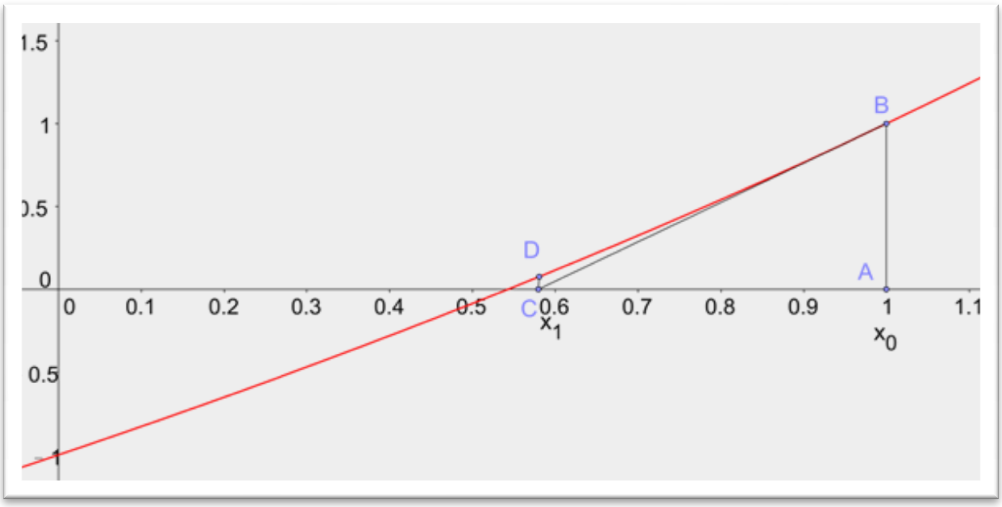
Since $f\,'(x) = \ln(2).2^x + 1$, the iterative formula is therefore: $x_n = x_{n-1} - \frac{2^{x_{n-1}} + x_{n-1} - 2}{\ln(2).2^{x_{n-1}} + 1}$

and we choose $x_0 = 1$ as the starting point.

Step	x
1	1
2	0.580940216
3	0.543252173
4	0.543000452
5	0.543000441
6	0.543000441

$\therefore$ The root of $\mathbf{2^x + x - 2 = 0}$ to be $x \approx \underline{\underline{\mathbf{0.5430004}}}$, to 7 decimal places in 6 steps.

Newton's method assumes the given function f to have a continuous derivative. The method may not converge if started too far away from a root. However, when it does converge, it is faster than the bisection method.



The above graph shows a little bit of how the Newton's method works. The red line denotes the function $\mathbf{y = f(x) = 2^x + x - 2}$. At A, the starting point $x_0 = 1$, we draw a vertical line which cuts the function at B. From B, we draw a tangent line BC which cuts the x-axis again at x_1. We continue the same way to get point D and then x_2 (not shown) The gradient of the tangent BC is $f'(x_0)$ which is equal to $\frac{BA}{AC}$.

$$f'(x_0) = \frac{BA}{AC} = \frac{f(x_0)}{x_0 - x_1}$$

We get $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. On deduction we get the **Newton-Raphson formula:** $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

(3) Secant Method

The secant method is defined by the recurrence relation

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$$

As can be seen from the recurrence relation, the secant method requires two initial values, x_0 and x_1, which should ideally be chosen to lie close to the root.

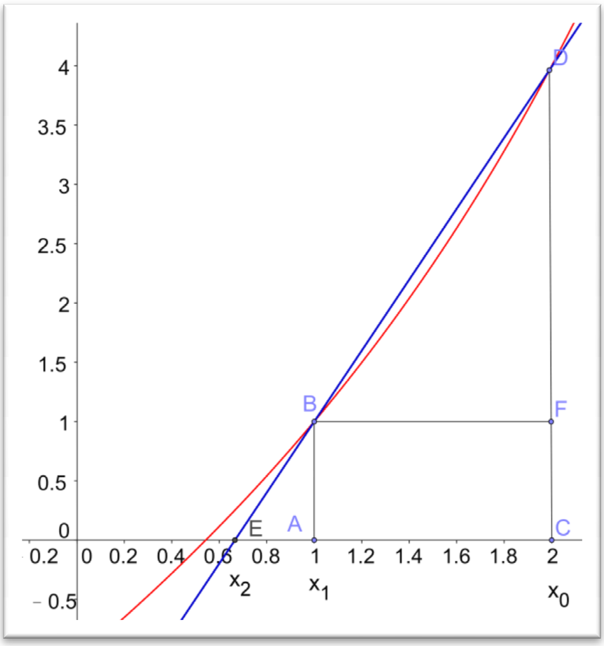
We choose $x_0 = 1, x_1 = 2$ as our starting values.

$$x_n = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})} = \frac{x_{n-2}(2^{x_{n-1}} + x_{n-1} - 2) - x_{n-1}(2^{x_{n-2}} + x_{n-2} - 2)}{(2^{x_{n-1}} + x_{n-1} - 2) - (2^{x_{n-2}} + x_{n-2} - 2)}$$

Step	x(n-1)	x(n-2)	x(n)
1	1	2	0.666666667
2	2	0.666666667	0.576233383
3	0.666666667	0.576233383	0.543722337
4	0.576233383	0.543722337	0.543004627
5	0.543722337	0.543004627	0.543000441
6	0.543004627	0.543000441	0.543000441

$\therefore$ The root of $\mathbf{2^x + x - 2 = 0}$ to be $x \approx \underline{\underline{\mathbf{0.5430004}}}$, to 7 decimal places in 6 steps.

The diagram shows how the secant method works:



The red line denotes the function $\mathbf{y = f(x) = 2^x + x - 2}$.

At A and C, the starting points $x_0 = 2, x_1 = 1$. DB is the secant or chord which meets the x-axis again at x_2. Since $\Delta EAB \sim \Delta BFD$, we have

$$\frac{AE}{BA} = \frac{FB}{DF} \Longrightarrow \frac{x_1 - x_2}{f(x_1)} = \frac{x_0 - x_1}{f(x_0) - f(x_1)} \Longrightarrow \frac{x_2 - x_1}{-f(x_1)} = \frac{x_1 - x_0}{f(x_1) - f(x_0)} \Longrightarrow x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

Now, we get a new set of initial points x_1, x_2 and we draw another secant line, deductively, we get:

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

(4) Fixed Point Iteration Method

Fixed Point Iteration Method

Given $f(x) = 0$

Convert the equation in the form $x = g(x)$.

Let the initial value be x_0 and use the iterative formula $x_{n+1} = g(x_n)$

Stop at $|x_{n+1} - g(x_n)| < \epsilon$, some tolerance limit.

Convergence condition: $|g'(x)| < 1$ for all x in an interval near the root of $f(x) = 0$.

You can make quite a lot of iterative formulas, but most of them do not converge:

1. $2^x + x - 2 = 0 \Longrightarrow x = 2 - 2^x \Longrightarrow x_{n+1} = 2 - 2^{x_n}$

2. $2^x + x - 2 = 0 \Longrightarrow 2^x = 2 - x \Longrightarrow x \ln 2 = \ln(2 - x) \Longrightarrow x_{n+1} = \frac{\ln(2 - x_n)}{\ln 2}$

3. $2^x + x - 2 = 0 \Longrightarrow x = 2^x + 2x - 2 \Longrightarrow x_{n+1} = 2^{x_n} + 2x_n - 2$

Since we know the root x_0, such that $0 < x_0 < 1$, we use the mid-point as initial value $x_0 = 0.5$.

Tests using EXCEL confirms that the iterative formulas in 1 and 3 cannot work and in 2, the iterative formula converges very slowly and after more than one thousand steps we get our root $x \approx \mathbf{0.5430004}$

Although **Fixed Point Iteration Method** does not work well in this example, it is a good method for other examples.

D. Lambert W-function

This problem can also be solved using Lambert W-function. The W-function is defined as the inverse of

$$f(x) = xe^x$$

that is, $x = W(xe^x)$ and $z = W(z)e^{W(z)}$.

Now, $2^x + x - 2 = 0 \Longrightarrow 2^x = 2 - x \Longrightarrow 2^{2-t} = t$, where $t = 2 - x$

So, $\frac{2^2}{2^t} = t \Longrightarrow 4 = t \times 2^t = t \times e^{t\log 2} \Longrightarrow 4\log 2 = (t\log 2) \times e^{t\log 2}$

Hence $W(4\log 2) = t\log 2 \Longrightarrow t = \frac{W(4\log 2)}{\log 2}$

Finally, we have $x = 2 - \frac{W(4\log 2)}{\log 2} = \frac{2\log 2 - W(4\log 2)}{\log 2} \approx \mathbf{0.543}$.

Exercise Find the root x_0 of $2^x = x^2$ such that $-1 < x_0 < 0$.
(Note that we omit the two other real roots x = 2 , 4.)